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Bi-derivation on Polynomial Ring

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Abstract
Derivations and their generalizations play an important role in understanding the structure of rings. One natural extension of derivations is the notion of a biderivation, which is a bi-additive mapping satisfying derivation-type identities in each argument. In this paper, we investigate biderivations on polynomial rings. Several examples of biderivations are constructed and some of their fundamental properties are established. In particular, we study the relationship between biderivations defined on a ring R and those induced on the polynomial ring $R[x]$. The results show that the set of bi-derivations on a polynomial ring is closed under finite sums, linear combinations, and finite direct sums. Moreover, the set of constants of a bi-derivation forms a $\mathbb{Z}$-module.

Keywords: *ring, polynomial ring, derivation, biderivation.*

1. Introduction

Derivations constitute an important tool in the structural study of rings and have been widely investigated in ring theory. A derivation on a ring R is an additive mapping $d : R \to R$ satisfying the Leibniz rule

$$d(ab) = d(a)b + ad(b) \tag{1}$$

for all $a, b \in R$. This concept can be regarded as an algebraic analogue of differentiation and has proved useful for describing structural properties of rings and their extensions. A comprehensive overview of several types of derivations and their developments in ring theory can be found in the survey by Ali et al. [1]. Classical results concerning derivations on prime rings were initiated in the work of Herstein [2], which has motivated many subsequent investigations.

Because of their structural importance, numerous generalizations of derivations have been introduced. Among these generalizations are biderivations, which extend the notion of derivations to mappings involving two variables and satisfying derivation-type identities in each argument. Various aspects of derivations and related additive mappings in rings have been studied in different algebraic contexts. For instance, Dhara and Sharma [3] examined

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additive mappings on rings with identity, while Ashraf et al. [4] introduced generalized (σ, τ)-biderivations and investigated their relationships with centralizers and bimultipliers.

Further investigations have focused on derivation-type mappings in special classes of rings. Kuzucuoğlu [5] studied Jordan derivations on strictly triangular matrix rings, and Kuzucuoğlu and Sayin [6] examined derivations on several classes of matrix rings. In prime and semiprime rings, Shujat [7] investigated symmetric generalized biderivations, while Reddy and Reddy [8] analyzed commutativity conditions for prime rings admitting symmetric biderivations. Related developments include studies of factor rings with derivations [9] and rings involving Lie and Jordan derivations [10, 11]. Additional results on Jordan derivations in matrix rings were obtained by Sayin and Kuzucuoğlu [12].

Recent work has continued to explore derivation-type mappings in broader algebraic settings. Ayupov and Yusupov [13] investigated 2-local derivations on infinite-dimensional Lie algebras, while De Filippis et al. [14] studied generalized g-derivations on prime rings. Other contributions include conditions under which symmetric biderivations force commutativity in prime rings [15], investigations of symmetric generalized biderivations associated with prime ideals [16], and stability problems for bi-derivations and bihomomorphisms in Banach algebras [17]. More recently, Murty and Reddy [18] examined orthogonal generalized symmetric reverse biderivations on semiprime rings, and Sahin and Kilic [19] studied skew symmetric Jordan biderivations on prime rings.

In addition to these studies, several authors have investigated derivations and related mappings in different algebraic constructions such as modules, group rings, and polynomial-type structures. Thomas et al. [20] studied derivations on several classes of rings. Fitriani et al. [21] introduced f-derivations on polynomial modules, while further developments on commuting and centralizing mappings on modules were obtained in [22]. Derivations and linear mappings on skew generalized power series modules were studied in [23], and (σ, τ)-derivations on group rings were investigated in [24]. Moreover, Jordan derivations on polynomial rings were considered by Sitompul et al. [25]. Other studies concerning derivations on polynomial rings include the investigation of nil derivations and δ-ideals in polynomial rings by Mursyidah et al. [26], as well as the study of (α', β')-derivations on polynomial rings by Syaharani et al. [27].

Polynomial rings themselves form one of the most fundamental constructions in ring theory. They provide a natural extension of a ring and often reveal additional structural properties of algebraic mappings. Despite the extensive literature on derivations and their generalizations, most existing works concentrate on prime rings, semiprime rings, matrix rings, or module-related structures. Comparatively less attention has been given to the study of biderivations in polynomial ring settings.

Motivated by these developments, the aim of this paper is to investigate biderivations on polynomial rings. In particular, we examine how biderivations defined on a ring R behave when extended to the polynomial ring $R[x]$. Unlike the case of a base ring R, the study of biderivations on the polynomial ring $R[x]$ involves additional technical challenges due to the presence of the indeterminate x. In particular, the behavior of $\lambda(x, r)$, for $r \in R$, plays a crucial role in determining how a biderivation extends from R to $R[x]$. Moreover, we construct several examples of biderivations and analyze their structural properties in the polynomial context. The results obtained in this paper provide further insight into the behavior of biderivations under polynomial extensions and contribute to the broader study of derivation-type mappings in ring theory.

2. Preliminaries

In this section we recall several basic concepts from ring theory that are needed throughout this paper. These notions provide the algebraic framework for studying biderivations and their behavior on polynomial rings. In this paper, unless otherwise specified, R is an arbitrary ring; commutativity is assumed only where explicitly indicated.

We begin with the definition of a ring, which serves as the fundamental algebraic structure in this work.

Definition 2.1. [28] *A ring R is an ordered triple $\langle R, +, \cdot \rangle$ consisting of a nonempty set R together with two binary operations, addition and multiplication, satisfying the following conditions:*

(1) $\langle R, + \rangle$ is an Abelian group;
(2) multiplication is associative;
(3) multiplication is distributive over addition, that is,

$$u(v + w) = uv + uw \tag{2}$$

and

$$(u + v)w = uw + vw \tag{3}$$

for all $u, v, w \in R$.

One of the most important constructions associated with a ring is the polynomial ring. Polynomial rings extend a given ring by introducing an indeterminate and play an important role in the study of algebraic structures. In particular, polynomial extensions allow us to investigate how algebraic mappings defined on a ring behave when extended to polynomial expressions.

Definition 2.2. [29] *Let R be a ring. The polynomial ring over R in the indeterminate x is denoted by $R[x]$. An element $\omega(x) \in R[x]$ can be written in the form*

$$\omega(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_q x^q, \tag{4}$$

where $q \geq 0$ and $r_k \in R$ for $k = 0, 1, \ldots, q$. If $\omega(x) = \sum_{k=0}^{q} r_k x^k$ and $\theta(x) = \sum_{k=0}^{p} b_k x^k$ belong to $R[x]$, then addition and multiplication are defined by

$$\omega(x) + \theta(x) = \sum_{k=0}^{\max(p,q)} (r_k + b_k) x^k \tag{5}$$

and

$$\omega(x)\theta(x) = \sum_{k=0}^{p+q} c_k x^k, \tag{6}$$

where

$$c_k = \sum_{j=0}^{k} r_j b_{k-j}. \tag{7}$$

Next we recall the notion of derivation, which can be regarded as an algebraic analogue of differentiation. Derivations play an important role in the structural study of rings and serve as a basis for several generalizations, including biderivations.

Definition 2.3. [1] *Let R be a ring. A mapping $\delta : R \to R$ is called a derivation if*

$$\delta(r + s) = \delta(r) + \delta(s), \tag{8}$$

and

$$\delta(rs) = \delta(r)s + r\delta(s) \tag{9}$$

for all $r, s \in R$.

To extend the concept of derivation to mappings involving two variables, we first introduce the notion of a bi-additive mapping.

Definition 2.4. [8] *Let R be a ring. A mapping $\lambda : R \times R \to R$ is called bi-additive if*

$$\lambda(r+s,t) = \lambda(r,t) + \lambda(s,t) \tag{10}$$

and

$$\lambda(r,s+t) = \lambda(r,s) + \lambda(r,t) \tag{11}$$

for all $r,s,t \in R$.

Bi-additive mappings that satisfy derivation-type identities in each argument are called biderivations. This notion can be viewed as a natural two-variable generalization of derivations.

Definition 2.5. [8] *A bi-additive mapping $\lambda : R \times R \to R$ is called a biderivation if*

$$\lambda(uv,w) = \lambda(u,w)v + u\lambda(v,w) \tag{12}$$

and

$$\lambda(u,vw) = \lambda(u,v)w + v\lambda(u,w) \tag{13}$$

for all $u,v,w \in R$.

These definitions provide the basic framework for the study carried out in the next section. In particular, we investigate how biderivations defined on a ring R can be extended to the polynomial ring $R[x]$ and analyze several structural properties of biderivations arising from this polynomial ring construction.

3. Results and Discussion

3.1. **Bi-Derivations on Rings.** To illustrate the concept of bi-derivations, we first present an example of a bi-derivation defined on a polynomial ring. This example demonstrates how the ordinary derivative induces a natural bi-derivation structure.

Example 3.1. *Let $R[x]$ be a polynomial ring. Define a mapping $\lambda : R[x] \times R[x] \to R[x]$ by*

$$\lambda(\omega,\theta) = \omega'\theta', \tag{14}$$

where ω' and θ' denote the ordinary derivatives of ω and θ with respect to x. We show that λ is a bi-derivation.
Let $\omega,\theta,\pi \in R[x]$. First, we verify that λ is bi-additive.

(1) $\lambda(\omega+\pi,\theta) = (\omega+\pi)'\theta' = (\omega'+\pi')\theta' = \omega'\theta' + \pi'\theta' = \lambda(\omega,\theta) + \lambda(\pi,\theta)$.
(2) $\lambda(\omega,\theta+\pi) = \omega'(\theta+\pi)' = \omega'(\theta'+\pi') = \omega'\theta' + \omega'\pi' = \lambda(\omega,\theta) + \lambda(\omega,\pi)$.

Hence, λ is bi-additive.
Next, we verify the derivation properties.

(1)

$$\begin{aligned}\lambda(\omega\pi,\theta) &= (\omega\pi)'\theta' \\ &= (\omega'\pi + \omega\pi')\theta' \\ &= \omega'\pi\theta' + \omega\pi'\theta'.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda(\omega,\theta)\pi + \omega\lambda(\pi,\theta) &= (\omega'\theta')\pi + \omega(\pi'\theta') \\ &= \omega'\theta'\pi + \omega\pi'\theta' \\ &= \omega'\pi\theta' + \omega\pi'\theta'.\end{aligned}$$

Thus,

$$\lambda(\omega\pi,\theta) = \lambda(\omega,\theta)\pi + \omega\lambda(\pi,\theta).$$

(2)

$$\begin{aligned}\lambda(\omega,\theta\pi) &= \omega'(\theta\pi)' \\ &= \omega'(\theta'\pi+\theta\pi') \\ &= \omega'\theta'\pi+\omega'\theta\pi'.\end{aligned}$$

Meanwhile,

$$\begin{aligned}\lambda(\omega,\theta)\pi+\theta\lambda(\omega,\pi) &= (\omega'\theta')\pi+\theta(\omega'\pi') \\ &= \omega'\theta'\pi+\theta\omega'\pi' \\ &= \omega'\theta'\pi+\omega'\theta\pi'.\end{aligned}$$

Therefore,

$$\lambda(\omega,\theta\pi) = \lambda(\omega,\theta)\pi+\theta\lambda(\omega,\pi).$$

Consequently, $\lambda(\omega,\theta)=\omega'\theta'$ satisfies all the properties of a bi-derivation. Hence, λ is a bi-derivation on $R[x]$.

We now present another example of a bi-derivation on a ring R.

Example 3.2. *Let $R=\mathbb{Z}_2[x]/\langle x^2\rangle$ and define*

$$\lambda(a+bx,\ c+dx) = (ad+bc)x.$$

We will show that λ is a bi-derivation on R.

(1) Let $u_1=a_1+b_1x$, $u_2=a_2+b_2x$, and $v=c+dx$. Then

$$u_1+u_2 = (a_1+a_2)+(b_1+b_2)x.$$

Hence,

$$\begin{aligned}\lambda(u_1+u_2,v) &= ((a_1+a_2)d+(b_1+b_2)c)x \\ &= (a_1d+b_1c)x+(a_2d+b_2c)x \\ &= \lambda(u_1,v)+\lambda(u_2,v).\end{aligned}$$

Similarly, λ is additive in the second variable. Thus, λ is bi-additive.

(2) Let $u=a+bx$, $v=c+dx$, and $w=e+fx$. Then

$$uv = (a+bx)(c+dx) = ac+(ad+bc)x,$$

since $x^2=0$. Thus,

$$\begin{aligned}\lambda(uv,w) &= \lambda(ac+(ad+bc)x,\ e+fx) \\ &= \big(acf+(ad+bc)e\big)x.\end{aligned}$$

On the other hand,

$$\lambda(u,w) = (af+be)x, \quad \lambda(v,w) = (cf+de)x.$$

Hence,

$$\begin{aligned}\lambda(u,w)v &= (af+be)x(c+dx) = (af+be)c\,x, \\ u\lambda(v,w) &= (a+bx)(cf+de)x = a(cf+de)x.\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda(u,w)v+u\lambda(v,w) &= \big((af+be)c+a(cf+de)\big)x \\ &= \big(afc+bec+acf+ade\big)x.\end{aligned}$$

Since we are in $\mathbb{Z}_2$, $afc+acf=0$, so this simplifies to

$$(bec+ade)x = \big(acf+(ad+bc)e\big)x,$$

which equals $\lambda(uv,w)$.

(3) A similar computation shows that $\lambda(u,vw)=\lambda(u,v)w+v\lambda(u,w)$.

Hence, λ is a bi-derivation on R.

Next, we construct a bi-derivation obtained from the sum of several bi-derivations.

Lemma 3.3. *Let R be a ring and let λ_k be a bi-derivation on R for each $k = 1, 2, \ldots, q$. Define*

$$\left(\sum_{k=1}^{q} \lambda_k\right) : R \times R \to R$$

by

$$\left(\sum_{k=1}^{q} \lambda_k\right)(r, s) = \lambda_1(r, s) + \lambda_2(r, s) + \cdots + \lambda_q(r, s). \tag{15}$$

Then $\sum_{k=1}^{q} \lambda_k$ is a bi-derivation on R.

Proof. Let $r, s, t \in R$. First, we verify that $\sum_{k=1}^{q} \lambda_k$ is bi-additive.

(1)

$$\begin{aligned}
\left(\sum_{k=1}^{q} \lambda_k\right)(r + s, t) &= \lambda_1(r + s, t) + \cdots + \lambda_q(r + s, t) \\
&= (\lambda_1(r, t) + \lambda_1(s, t)) + \cdots + (\lambda_q(r, t) + \lambda_q(s, t)) \\
&= \left(\sum_{k=1}^{q} \lambda_k(r, t)\right) + \left(\sum_{k=1}^{q} \lambda_k(s, t)\right) \\
&= \left(\sum_{k=1}^{q} \lambda_k\right)(r, t) + \left(\sum_{k=1}^{q} \lambda_k\right)(s, t).
\end{aligned}$$

(2)

$$\begin{aligned}
\left(\sum_{k=1}^{q} \lambda_k\right)(r, s + t) &= \lambda_1(r, s + t) + \cdots + \lambda_q(r, s + t) \\
&= (\lambda_1(r, s) + \lambda_1(r, t)) + \cdots + (\lambda_q(r, s) + \lambda_q(r, t)) \\
&= \left(\sum_{k=1}^{q} \lambda_k(r, s)\right) + \left(\sum_{k=1}^{q} \lambda_k(r, t)\right) \\
&= \left(\sum_{k=1}^{q} \lambda_k\right)(r, s) + \left(\sum_{k=1}^{q} \lambda_k\right)(r, t).
\end{aligned}$$

Thus, $\sum_{k=1}^{q} \lambda_k$ is bi-additive.

Next, we verify the derivation properties.

(1)

$$\begin{aligned}
\left(\sum_{k=1}^{q} \lambda_k\right)(rs, t) &= \lambda_1(rs, t) + \cdots + \lambda_q(rs, t) \\
&= (\lambda_1(r, t)s + r\lambda_1(s, t)) + \cdots + (\lambda_q(r, t)s + r\lambda_q(s, t)) \\
&= \left(\sum_{k=1}^{q} \lambda_k(r, t)\right) s + r\left(\sum_{k=1}^{q} \lambda_k(s, t)\right) \\
&= \left(\sum_{k=1}^{q} \lambda_k\right)(r, t)s + r\left(\sum_{k=1}^{q} \lambda_k\right)(s, t).
\end{aligned}$$

(2)

$$\begin{aligned}\left(\sum_{k=1}^{q}\lambda_k\right)(r,st) &= \lambda_1(r,st)+\cdots+\lambda_q(r,st)\\ &= (\lambda_1(r,s)t+s\lambda_1(r,t))+\cdots+(\lambda_q(r,s)t+s\lambda_q(r,t))\\ &= \left(\sum_{k=1}^{q}\lambda_k(r,s)\right)t+s\left(\sum_{k=1}^{q}\lambda_k(r,t)\right)\\ &= \left(\sum_{k=1}^{q}\lambda_k\right)(r,s)t+s\left(\sum_{k=1}^{q}\lambda_k\right)(r,t).\end{aligned}$$

Therefore, $\sum_{k=1}^{q}\lambda_k$ is a bi-derivation on R. □

The following example illustrates that the sum of two bi-derivations is again a bi-derivation.

Example 3.4. *Let*

$$M=\left\{\begin{bmatrix}0 & \mu\\ 0 & \tau\end{bmatrix}\,\middle|\, \mu,\tau\in\mathbb{Z}\right\}$$

be a ring under usual matrix addition and multiplication.
Define mappings $\lambda_1,\lambda_2 : M\times M\to M$ *by*

$$\lambda_1(K,L)=KL-LK$$

and

$$\lambda_2(K,L)=\begin{bmatrix}0 & \mu_1\mu_2\\ 0 & 0\end{bmatrix},$$

where

$$K=\begin{bmatrix}0 & \mu_1\\ 0 & \tau_1\end{bmatrix},\quad L=\begin{bmatrix}0 & \mu_2\\ 0 & \tau_2\end{bmatrix}.$$

It can be verified that both λ_1 *and* λ_2 *satisfy the defining properties of a bi-derivation on* M*.*
Let

$$K=\begin{bmatrix}0 & \mu_1\\ 0 & \tau_1\end{bmatrix},\quad L=\begin{bmatrix}0 & \mu_2\\ 0 & \tau_2\end{bmatrix},\quad N=\begin{bmatrix}0 & \mu_3\\ 0 & \tau_3\end{bmatrix}\in M.$$

We verify that $\lambda_1+\lambda_2$ *satisfies the bi-derivation properties.*
First, we check bi-additivity.

(1)

$$(\lambda_1+\lambda_2)(K+L,N)=\lambda_1(K+L,N)+\lambda_2(K+L,N).$$

Using the bi-additivity of λ_1 *and* λ_2*, we obtain*

$$(\lambda_1+\lambda_2)(K+L,N)=(\lambda_1+\lambda_2)(K,N)+(\lambda_1+\lambda_2)(L,N).$$

(2) Similarly,

$$(\lambda_1+\lambda_2)(K,L+N)=(\lambda_1+\lambda_2)(K,L)+(\lambda_1+\lambda_2)(K,N).$$

Next, we verify the derivation identities.

(1)

$$(\lambda_1+\lambda_2)(KL,N)=\lambda_1(KL,N)+\lambda_2(KL,N).$$

Using the bi-derivation property of λ_1 *and* λ_2*, we obtain*

$$(\lambda_1+\lambda_2)(KL,N)=(\lambda_1+\lambda_2)(K,N)L+K(\lambda_1+\lambda_2)(L,N).$$

(2) Similarly,

$$(\lambda_1+\lambda_2)(K,LN)=(\lambda_1+\lambda_2)(K,L)N+L(\lambda_1+\lambda_2)(K,N).$$

Therefore, $\lambda_1+\lambda_2$ *satisfies all the properties of a bi-derivation. Hence,* $\lambda_1+\lambda_2$ *is a bi-derivation on* M*.*

Furthermore, linear combinations of bi-derivations provide a natural way to construct new bi-derivations from existing ones. The following proposition shows that the set of bi-derivations is closed under linear combinations.

Proposition 3.5. *Let R be a ring and let $\lambda_1, \lambda_2, \dots, \lambda_p$ be bi-derivations on R. For $\ell_1, \ell_2, \dots, \ell_p \in R$, define a mapping*

$$\Lambda : R \times R \to R$$

by

$$\Lambda(r, s) = \sum_{k=1}^{p} \ell_k \lambda_k(r, s) \tag{16}$$

for all $r, s \in R$. Then Λ is a bi-derivation on R.

Proof. Let $r, s, t \in R$. First, we verify that Λ is bi-additive. Since each λ_k is bi-additive, we have

$$\lambda_k(r + s, t) = \lambda_k(r, t) + \lambda_k(s, t) \tag{17}$$

and

$$\lambda_k(r, s + t) = \lambda_k(r, s) + \lambda_k(r, t). \tag{18}$$

Hence

$$\Lambda(r + s, t) = \sum_{k=1}^{p} \ell_k \lambda_k(r + s, t) = \sum_{k=1}^{p} \ell_k \big(\lambda_k(r, t) + \lambda_k(s, t)\big)$$

$$= \sum_{k=1}^{p} \ell_k \lambda_k(r, t) + \sum_{k=1}^{p} \ell_k \lambda_k(s, t) = \Lambda(r, t) + \Lambda(s, t).$$

Similarly,

$$\Lambda(r, s + t) = \Lambda(r, s) + \Lambda(r, t).$$

Next, we verify the derivation identities. Since each λ_k is a bi-derivation,

$$\lambda_k(rs, t) = \lambda_k(r, t)s + r\lambda_k(s, t).$$

Thus

$$\Lambda(rs, t) = \sum_{k=1}^{p} \ell_k \lambda_k(rs, t)$$

$$= \sum_{k=1}^{p} \ell_k \big(\lambda_k(r, t)s + r\lambda_k(s, t)\big)$$

$$= \left(\sum_{k=1}^{p} \ell_k \lambda_k(r, t)\right) s + r \left(\sum_{k=1}^{p} \ell_k \lambda_k(s, t)\right)$$

$$= \Lambda(r, t)s + r\Lambda(s, t).$$

Similarly,

$$\Lambda(r, st) = \Lambda(r, s)t + s\Lambda(r, t). \tag{19}$$

Therefore, Λ satisfies all defining properties of a bi-derivation on R. □

The following example illustrates Proposition 3.5

Example 3.6. *Let*

$$M = \left\{ \begin{bmatrix} \mu & 0 \\ \tau & 0 \end{bmatrix} \middle| \mu, \tau \in \mathbb{Z} \right\}.$$

Define mappings $\lambda_1, \lambda_2 : M \times M \to M$ by

$$\lambda_1(K, L) = KL - LK$$

and

$$\lambda_2(K, L) = \begin{bmatrix} 0 & 0 \\ \tau_1\mu_2 & 0 \end{bmatrix},$$

where

$$K = \begin{bmatrix} \mu_1 & 0 \\ \tau_1 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} \mu_2 & 0 \\ \tau_2 & 0 \end{bmatrix}.$$

It can be verified that λ_1 and λ_2 are bi-derivations on M. By Proposition 3.5, any linear combination

$$\ell_1\lambda_1 + \ell_2\lambda_2$$

with $\ell_1, \ell_2 \in M$ is also a bi-derivation on M.

Proposition 3.5 shows that linear combinations of bi-derivations again yield a bi-derivation. Consequently, the collection of all bi-derivations on a ring possesses a natural algebraic structure under pointwise addition.

Corollary 3.7. *Let R be a ring. The set of all bi-derivations on R forms an Abelian group under pointwise addition.*

The following example illustrates the Abelian group structure of the set of bi-derivations.

Example 3.8. *Let*

$$M = \left\{ \begin{bmatrix} \mu & 0 \\ \tau & 0 \end{bmatrix} \;\middle|\; \mu, \tau \in \mathbb{Z} \right\}.$$

Define $\lambda : M \times M \to M$ by

$$\lambda(K, L) = \begin{bmatrix} 0 & 0 \\ \tau_1\mu_2 & 0 \end{bmatrix},$$

where

$$K = \begin{bmatrix} \mu_1 & 0 \\ \tau_1 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} \mu_2 & 0 \\ \tau_2 & 0 \end{bmatrix}.$$

It can be verified that λ is a bi-derivation on M. Define the mapping $-\lambda : M \times M \to M$ by

$$(-\lambda)(K, L) = -\lambda(K, L).$$

Since the additive inverse exists in M, it follows immediately that $-\lambda$ satisfies the defining properties of a bi-derivation. Hence $-\lambda$ is also a bi-derivation on M. This example illustrates the existence of additive inverses in the Abelian group of bi-derivations.

Remark 3.9. *The previous result shows that the set of all bi-derivations on R forms an Abelian group under pointwise addition.*

After discussing the definition and basic properties of bi-derivations, the discussion can be extended to more complex algebraic constructions. One such construction is the direct sum of bi-derivations, which arises naturally from the direct sum of rings.

Proposition 3.10. *Let $R_1, R_2, \ldots, R_p$ be rings and let $\lambda_k : R_k \times R_k \to R_k$ be mappings for each $k = 1, 2, \ldots, p$. Define the mapping*

$$\bigoplus_{k=1}^{p} \lambda_k : \left(\bigoplus_{k=1}^{p} R_k\right) \times \left(\bigoplus_{k=1}^{p} R_k\right) \to \bigoplus_{k=1}^{p} R_k$$

by

$$\left(\bigoplus_{k=1}^{p} \lambda_k\right) \big((u_1, \ldots, u_p), (v_1, \ldots, v_p)\big) = (\lambda_1(u_1, v_1), \ldots, \lambda_p(u_p, v_p)). \tag{20}$$

If each λ_k is a bi-derivation on R_k, then $\bigoplus_{k=1}^{p} \lambda_k$ is a bi-derivation on $\bigoplus_{k=1}^{p} R_k$.

Proof. Let $u = (u_1, \ldots, u_p)$, $v = (v_1, \ldots, v_p)$, and $w = (w_1, \ldots, w_p)$ be elements of $\bigoplus_{k=1}^{p} R_k$. Since addition and multiplication in the direct sum are defined componentwise, we have

$$(u + v)_k = u_k + v_k, \qquad (uv)_k = u_k v_k$$

for each $k = 1, \dots, p$. First,

$$\begin{aligned}(\bigoplus_{k=1}^{p} \lambda_k)(u+v, w) &= (\lambda_1(u_1+v_1, w_1), \dots, \lambda_p(u_p+v_p, w_p)) \\ &= (\lambda_1(u_1, w_1) + \lambda_1(v_1, w_1), \dots, \lambda_p(u_p, w_p) + \lambda_p(v_p, w_p)) \\ &= (\bigoplus_{k=1}^{p} \lambda_k)(u, w) + (\bigoplus_{k=1}^{p} \lambda_k)(v, w).\end{aligned}$$

Similarly,

$$(\bigoplus_{k=1}^{p} \lambda_k)(u, v+w) = (\bigoplus_{k=1}^{p} \lambda_k)(u, v) + (\bigoplus_{k=1}^{p} \lambda_k)(u, w).$$

Next,

$$\begin{aligned}(\bigoplus_{k=1}^{p} \lambda_k)(uv, w) &= (\lambda_1(u_1v_1, w_1), \dots, \lambda_p(u_pv_p, w_p)) \\ &= (\lambda_1(u_1, w_1)v_1 + u_1\lambda_1(v_1, w_1), \dots, \lambda_p(u_p, w_p)v_p + u_p\lambda_p(v_p, w_p)) \\ &= (\bigoplus_{k=1}^{p} \lambda_k)(u, w)v + u(\bigoplus_{k=1}^{p} \lambda_k)(v, w).\end{aligned}$$

Likewise,

$$(\bigoplus_{k=1}^{p} \lambda_k)(u, vw) = (\bigoplus_{k=1}^{p} \lambda_k)(u, v)w + v(\bigoplus_{k=1}^{p} \lambda_k)(u, w).$$

Thus $\bigoplus_{k=1}^{p} \lambda_k$ satisfies all defining properties of a bi-derivation. Therefore $\bigoplus_{k=1}^{p} \lambda_k$ is a bi-derivation on $\bigoplus_{k=1}^{p} R_k$. □

The following example illustrates the direct sum of two bi-derivations.

Example 3.11. *Let*

$$M_1 = \left\{ \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \mid b, d \in \mathbb{Z} \right\}, \qquad M_2 = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \mid a, c \in \mathbb{Z} \right\}.$$

Define mappings $\lambda_1 : M_1 \times M_1 \to M_1$ and $\lambda_2 : M_2 \times M_2 \to M_2$ by

$$\lambda_1(X, Y) = \begin{bmatrix} 0 & b_1b_2 \\ 0 & 0 \end{bmatrix}, \qquad \lambda_2(U, V) = \begin{bmatrix} 0 & 0 \\ c_1a_2 & 0 \end{bmatrix},$$

for $X, Y \in M_1$ and $U, V \in M_2$. It is straightforward to verify that λ_1 and λ_2 are bi-derivations on M_1 and M_2, respectively. Now define

$$\lambda_1 \oplus \lambda_2 : (M_1 \oplus M_2) \times (M_1 \oplus M_2) \to M_1 \oplus M_2$$

by

$$(\lambda_1 \oplus \lambda_2)((K_1, K_2), (L_1, L_2)) = (\lambda_1(K_1, L_1), \lambda_2(K_2, L_2)).$$

For $(K_1, K_2), (L_1, L_2), (N_1, N_2) \in M_1 \oplus M_2$ we obtain

$$(\lambda_1 \oplus \lambda_2)((K_1, K_2) + (L_1, L_2), (N_1, N_2)) = (\lambda_1(K_1 + L_1, N_1), \lambda_2(K_2 + L_2, N_2)).$$

Since λ_1 and λ_2 are bi-additive, it follows that

$$\lambda_1 \oplus \lambda_2((K_1, K_2) + (L_1, L_2), (N_1, N_2)) = \lambda_1 \oplus \lambda_2((K_1, K_2), (N_1, N_2)) + \lambda_1 \oplus \lambda_2((L_1, L_2), (N_1, N_2)).$$

Similarly, one can verify the remaining bi-derivation identities. Hence, $\lambda_1 \oplus \lambda_2$ is a bi-derivation on $M_1 \oplus M_2$.

After discussing the direct sum of bi-derivations on rings, we now examine some fundamental properties of bi-derivations. In particular, we begin with the definition of the zero bi-derivation.

Definition 3.12. *Let R be a ring. Define a mapping*

$$0 : R \times R \to R$$

by $0(a,b) = 0_R$ for all $(a,b) \in R \times R$, where 0_R denotes the zero element of R. This mapping is called the zero bi-derivation on R.

Next, we study the algebraic structure formed by the set of all bi-derivations on a ring.

Theorem 3.13. *Let R be a ring and define*

$$\wp = \{\lambda : R \times R \to R \mid \lambda \text{ is a bi-derivation on } R\}.$$

Then $\wp$ is a $\mathbb{Z}$-module.

Proof. First we show that $(\wp, +)$ forms an Abelian group under pointwise addition.

(1) The zero mapping $0 : R \times R \to R$ defined by $0(a,b) = 0_R$ for all $(a,b) \in R \times R$ is a bi-derivation on R. Hence $0 \in \wp$, so $\wp \neq \emptyset$.
(2) Let $\lambda_1, \lambda_2 \in \wp$. By Lemma 3.3, the sum $\lambda_1 + \lambda_2$ is also a bi-derivation on R. Hence $\lambda_1 + \lambda_2 \in \wp$.
(3) The zero mapping 0 acts as the additive identity, since

$$\lambda + 0 = 0 + \lambda = \lambda$$

for every $\lambda \in \wp$.
(4) For each $\lambda \in \wp$, the mapping $-\lambda$ is also a bi-derivation by Corollary **??**. Hence

$$\lambda + (-\lambda) = (-\lambda) + \lambda = 0.$$

(5) For any $\lambda_1, \lambda_2 \in \wp$ and $(a,b) \in R \times R$,

$$(\lambda_1 + \lambda_2)(a,b) = \lambda_1(a,b) + \lambda_2(a,b) = \lambda_2(a,b) + \lambda_1(a,b) = (\lambda_2 + \lambda_1)(a,b).$$

Hence $\lambda_1 + \lambda_2 = \lambda_2 + \lambda_1$.

Thus $(\wp, +)$ is an Abelian group. Since scalar multiplication by integers can be defined by repeated addition, it follows that $\wp$ is a $\mathbb{Z}$-module. □

To further study the structure of bi-derivations on a ring R, we consider the set of constants associated with a bi-derivation. The following theorem shows that this set possesses a natural $\mathbb{Z}$-module structure.

Theorem 3.14. *Let R be a ring and let $\lambda : R \times R \to R$ be a bi-derivation. Define*

$$C_\lambda = \{(r,s) \in R \times R \mid \lambda(r,s) = 0\}. \tag{21}$$

Then C_λ is a $\mathbb{Z}$-module.

Proof. We show that C_λ is a $\mathbb{Z}$-submodule of $R \times R$.

(1) Let $(r_1, s_1), (r_2, s_2) \in C_\lambda$. Then

$$\lambda(r_1, s_1) = 0 \quad \text{and} \quad \lambda(r_2, s_2) = 0.$$

Since λ is bi-additive, we obtain

$$\lambda((r_1, s_1) + (r_2, s_2)) = \lambda(r_1 + r_2, s_1 + s_2) = \lambda(r_1, s_1) + \lambda(r_2, s_2) = 0.$$

Hence $(r_1 + r_2, s_1 + s_2) \in C_\lambda$.
(2) Let $(r,s) \in C_\lambda$. Since $\lambda(r,s) = 0$ and λ is additive, we have

$$\lambda(-r, -s) = -\lambda(r,s) = 0.$$

Thus $(-r, -s) \in C_\lambda$.
(3) Let $n \in \mathbb{Z}$ and $(r,s) \in C_\lambda$. Then

$$\lambda(nr, ns) = n\lambda(r,s) = n \cdot 0 = 0.$$

Hence $(nr, ns) \in C_\lambda$.

Therefore C_λ is closed under addition, additive inverses, and integer scalar multiplication. Hence C_λ is a $\mathbb{Z}$-module. □

Having established several structural properties of bi-derivations on rings, we now extend the discussion to polynomial rings. The properties of bi-derivations on R discussed above extend naturally to the polynomial ring $R[x]$.

3.2. **Bi-Derivation on Polynomial Ring.** A derivation on a polynomial ring is a mapping that assigns each polynomial to another polynomial and satisfies the additive property and the Leibniz rule. This concept can naturally be extended to bi-derivations on polynomial rings. The following definition introduces a bi-derivation on the polynomial ring $R[x]$.

Definition 3.15. *Let R be a ring and let $R[x]$ be the polynomial ring over R. A mapping*

$$\hat{\lambda} : R[x] \times R[x] \to R[x]$$

is called a bi-derivation on $R[x]$ if for all $\omega(x), \theta(x), \pi(x) \in R[x]$ the following conditions hold:

(1)

$$\hat{\lambda}(\omega(x) + \theta(x), \pi(x)) = \hat{\lambda}(\omega(x), \pi(x)) + \hat{\lambda}(\theta(x), \pi(x)),$$

$$\hat{\lambda}(\omega(x), \theta(x) + \pi(x)) = \hat{\lambda}(\omega(x), \theta(x)) + \hat{\lambda}(\omega(x), \pi(x)).$$

(2)

$$\hat{\lambda}(\omega(x)\theta(x), \pi(x)) = \hat{\lambda}(\omega(x), \pi(x))\theta(x) + \omega(x)\hat{\lambda}(\theta(x), \pi(x)),$$

$$\hat{\lambda}(\omega(x), \theta(x)\pi(x)) = \hat{\lambda}(\omega(x), \theta(x))\pi(x) + \theta(x)\hat{\lambda}(\omega(x), \pi(x)).$$

The following theorem shows that every bi-derivation on a ring R can be extended naturally to the polynomial ring $R[x]$.

Theorem 3.16. *Let R be a ring and let $\lambda : R \times R \to R$ be a bi-derivation on R. Then there exists a mapping*

$$\hat{\lambda} : R[x] \times R[x] \to R[x]$$

which is a bi-derivation on $R[x]$.

Proof. Let

$$\omega(x) = \sum_{k=0}^{p} r_k x^k, \qquad \theta(x) = \sum_{\ell=0}^{q} s_\ell x^\ell$$

be polynomials in $R[x]$. Define

$$\hat{\lambda}(\omega(x), \theta(x)) = \sum_{k=0}^{p} \sum_{\ell=0}^{q} \lambda(r_k, s_\ell) x^{k+\ell}.$$

First we show that $\hat{\lambda}$ is bi-additive. Let $\omega_1(x), \omega_2(x), \theta(x) \in R[x]$ where

$$\omega_1(x) = \sum_{k=0}^{p} r_k x^k, \qquad \omega_2(x) = \sum_{k=0}^{p} \tilde{r}_k x^k.$$

Then

$$\omega_1(x) + \omega_2(x) = \sum_{k=0}^{p} (r_k + \tilde{r}_k) x^k.$$

Thus

$$\begin{aligned}\hat{\lambda}(\omega_1(x)+\omega_2(x),\theta(x)) &= \sum_{k=0}^{p}\sum_{\ell=0}^{q}\lambda(r_k+\tilde{r}_k,s_\ell)x^{k+\ell}\\ &= \sum_{k=0}^{p}\sum_{\ell=0}^{q}\big(\lambda(r_k,s_\ell)+\lambda(\tilde{r}_k,s_\ell)\big)x^{k+\ell}\\ &= \hat{\lambda}(\omega_1(x),\theta(x))+\hat{\lambda}(\omega_2(x),\theta(x)).\end{aligned}$$

Similarly,

$$\hat{\lambda}(\omega(x),\theta_1(x)+\theta_2(x)) = \hat{\lambda}(\omega(x),\theta_1(x))+\hat{\lambda}(\omega(x),\theta_2(x)).$$

Hence $\hat{\lambda}$ is bi-additive. Now, let

$$\pi(x) = \sum_{m=0}^{s} t_m x^m.$$

Then

$$\omega(x)\theta(x) = \sum_{t=0}^{p+q}\left(\sum_{k+\ell=t} r_k s_\ell\right)x^t.$$

Thus

$$\hat{\lambda}(\omega(x)\theta(x),\pi(x)) = \sum_{t=0}^{p+q}\sum_{m=0}^{s}\lambda\left(\sum_{k+\ell=t} r_k s_\ell, t_m\right)x^{t+m}.$$

Since λ is a bi-derivation on R,

$$\lambda(r_k s_\ell, t_m) = \lambda(r_k,t_m)s_\ell + r_k\lambda(s_\ell,t_m).$$

Substituting this expression yields

$$\hat{\lambda}(\omega(x)\theta(x),\pi(x)) = \hat{\lambda}(\omega(x),\pi(x))\theta(x)+\omega(x)\hat{\lambda}(\theta(x),\pi(x)).$$

Similarly one can prove that

$$\hat{\lambda}(\omega(x),\theta(x)\pi(x)) = \hat{\lambda}(\omega(x),\theta(x))\pi(x)+\theta(x)\hat{\lambda}(\omega(x),\pi(x)).$$

Therefore $\hat{\lambda}$ is a bi-derivation on the polynomial ring $R[x]$. □

The construction of $\hat{\lambda}$ is not unique. Let μ be a bi-derivation on R and let

$$\omega(x) = \sum_{k=0}^{p} r_k x^k, \qquad \theta(x) = \sum_{\ell=0}^{q} s_\ell x^\ell$$

be polynomials in $R[x]$. We can define

$$\lambda'(\omega(x),\theta(x)) = \sum_{k=0}^{p}\sum_{\ell=0}^{q}\mu(r_k,s_\ell)x^{k+\ell}.$$

Then λ' defines a bi-derivation on $R[x]$.

The following examples illustrate bi-derivations defined on polynomial rings.

Example 3.17. *Let R be a commutative ring and let $R[x]$ be the polynomial ring over R. Define a mapping*

$$\hat{\lambda} : R[x]\times R[x]\to R[x]$$

by

$$\hat{\lambda}(\omega(x),\theta(x)) = a\,\omega'(x)\theta'(x),$$

where $\omega(x), \theta(x) \in R[x]$ and $a \in R$. We show that $\hat{\lambda}$ is a bi-derivation. First, $\hat{\lambda}$ is bi-additive. For $\omega(x), \theta(x), \pi(x) \in R[x]$,

$$\begin{aligned}\hat{\lambda}(\omega(x)+\theta(x),\pi(x)) &= a(\omega(x)+\theta(x))'\pi'(x)\\ &= a(\omega'(x)+\theta'(x))\pi'(x)\\ &= a\omega'(x)\pi'(x)+a\theta'(x)\pi'(x)\\ &= \hat{\lambda}(\omega(x),\pi(x))+\hat{\lambda}(\theta(x),\pi(x)).\end{aligned}$$

Similarly,

$$\hat{\lambda}(\omega(x),\theta(x)+\pi(x)) = \hat{\lambda}(\omega(x),\theta(x))+\hat{\lambda}(\omega(x),\pi(x)).$$

Next,

$$\begin{aligned}\hat{\lambda}(\omega(x)\theta(x),\pi(x)) &= a(\omega(x)\theta(x))'\pi'(x)\\ &= a(\omega'(x)\theta(x)+\omega(x)\theta'(x))\pi'(x)\\ &= \hat{\lambda}(\omega(x),\pi(x))\theta(x)+\omega(x)\hat{\lambda}(\theta(x),\pi(x)).\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{\lambda}(\omega(x),\theta(x)\pi(x)) &= a\omega'(x)(\theta(x)\pi(x))'\\ &= a\omega'(x)(\theta'(x)\pi(x)+\theta(x)\pi'(x))\\ &= \hat{\lambda}(\omega(x),\theta(x))\pi(x)+\theta(x)\hat{\lambda}(\omega(x),\pi(x)).\end{aligned}$$

Therefore $\hat{\lambda}$ is a bi-derivation on $R[x]$.

A concrete illustration of the above bi-derivation is given in the following example.

Example 3.18. *Let R be a commutative ring and define*

$$\hat{\lambda}(\omega(x),\theta(x)) = a\,\omega'(x)\theta'(x), \qquad a \in R.$$

Consider the polynomials

$$\omega(x) = x^2, \qquad \theta(x) = x+x^2, \qquad \pi(x) = 2x^3.$$

First compute

$$\omega'(x) = 2x, \qquad \theta'(x) = 1+2x, \qquad \pi'(x) = 6x^2.$$

Then

$$\hat{\lambda}(\omega(x),\pi(x)) = a(2x)(6x^2) = 12ax^3$$

and

$$\hat{\lambda}(\theta(x),\pi(x)) = a(1+2x)(6x^2) = 6ax^2+12ax^3.$$

Hence

$$\hat{\lambda}(\omega(x),\pi(x))+\hat{\lambda}(\theta(x),\pi(x)) = 6ax^2+24ax^3.$$

On the other hand,

$$\omega(x)+\theta(x) = x+2x^2,$$

so

$$(\omega(x)+\theta(x))' = 1+4x.$$

Thus

$$\hat{\lambda}(\omega(x)+\theta(x),\pi(x)) = a(1+4x)(6x^2) = 6ax^2+24ax^3.$$

Therefore

$$\hat{\lambda}(\omega(x)+\theta(x),\pi(x)) = \hat{\lambda}(\omega(x),\pi(x))+\hat{\lambda}(\theta(x),\pi(x)).$$

Hence $\hat{\lambda}$ defines a bi-derivation on $R[x]$.

The following result shows that the set of bi-derivations is closed under linear combinations.

Theorem 3.19. *Let $R[x]$ be a polynomial ring. If $\hat{\lambda}_k$ is a bi-derivation on $R[x]$ for each $k = 1, 2, \ldots, n$, and $a_k \in R$, then*

$$\sum_{k=1}^{n} a_k \hat{\lambda}_k$$

is also a bi-derivation on $R[x]$.

Proof. Let $\omega(x), \theta(x), \pi(x) \in R[x]$. Define

$$\Lambda(\omega(x), \theta(x)) = \sum_{k=1}^{n} a_k \hat{\lambda}_k(\omega(x), \theta(x)). \tag{22}$$

First we show that Λ is bi-additive. Since each $\hat{\lambda}_k$ is bi-additive, we have

$$\begin{aligned}\Lambda(\omega(x) + \theta(x), \pi(x)) &= \sum_{k=1}^{n} a_k \hat{\lambda}_k(\omega(x) + \theta(x), \pi(x)) \\ &= \sum_{k=1}^{n} a_k \big(\hat{\lambda}_k(\omega(x), \pi(x)) + \hat{\lambda}_k(\theta(x), \pi(x))\big) \\ &= \Lambda(\omega(x), \pi(x)) + \Lambda(\theta(x), \pi(x)).\end{aligned}$$

Similarly,

$$\Lambda(\omega(x), \theta(x) + \pi(x)) = \Lambda(\omega(x), \theta(x)) + \Lambda(\omega(x), \pi(x)). \tag{23}$$

Since each $\hat{\lambda}_k$ is a bi-derivation,

$$\begin{aligned}\Lambda(\omega(x)\theta(x), \pi(x)) &= \sum_{k=1}^{n} a_k \hat{\lambda}_k(\omega(x)\theta(x), \pi(x)) \\ &= \sum_{k=1}^{n} a_k \big(\hat{\lambda}_k(\omega(x), \pi(x))\theta(x) + \omega(x)\hat{\lambda}_k(\theta(x), \pi(x))\big) \\ &= \Lambda(\omega(x), \pi(x))\theta(x) + \omega(x)\Lambda(\theta(x), \pi(x)).\end{aligned}$$

Similarly,

$$\Lambda(\omega(x), \theta(x)\pi(x)) = \Lambda(\omega(x), \theta(x))\pi(x) + \theta(x)\Lambda(\omega(x), \pi(x)). \tag{24}$$

Hence Λ is a bi-derivation on $R[x]$. □

The following example illustrates Theorem above.

Example 3.20. *Let R be a commutative ring and let $R[x]$ be the polynomial ring over R. Define maps $\hat{\lambda}_1, \hat{\lambda}_2 : R[x] \times R[x] \to R[x]$ by*

$$\hat{\lambda}_1(\omega(x), \theta(x)) = \omega'(x)\theta'(x), \qquad \hat{\lambda}_2(\omega(x), \theta(x)) = a\,\omega'(x)\theta'(x),$$

where $a \in R$ and $\omega(x), \theta(x) \in R[x]$. It is straightforward to verify that both $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are bi-derivations on $R[x]$. Let $a_1, a_2 \in R$ and define

$$\Lambda = a_1\hat{\lambda}_1 + a_2\hat{\lambda}_2.$$

Then for any $\omega(x), \theta(x) \in R[x]$,

$$\Lambda(\omega(x), \theta(x)) = a_1\omega'(x)\theta'(x) + a_2 a\,\omega'(x)\theta'(x) = (a_1 + aa_2)\omega'(x)\theta'(x).$$

Hence Λ is again a bi-derivation on $R[x]$, which illustrates that linear combinations of bi-derivations remain bi-derivations.

4. Conclusion

In this paper, we studied the properties of bi-derivations on rings and their extensions to polynomial rings. It was shown that a bi-derivation is a mapping $\lambda : R \times R \to R$ that is bi-additive and satisfies the identities $\lambda(xy, z) = \lambda(x, z)y + x\lambda(y, z)$ and $\lambda(x, yz) = \lambda(x, y)z + y\lambda(x, z)$ for all $x, y, z \in R$.

Furthermore, we show that the set of all bi-derivations, together with the set of constants associated to a given bi-derivation, each forms a $\mathbb{Z}$-module. In the setting of polynomial rings $R[x]$, bi-derivations can be constructed by extending the bi-additive structure from the base ring R to the polynomial coefficients. Consequently, whenever a ring R admits a bi-derivation, the polynomial ring $R[x]$ also admits a bi-derivation with analogous structural properties. These results enhance the understanding of the algebraic behavior of bi-derivations and demonstrate that their key structural features are preserved under polynomial ring extensions.

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