

## Alternative Proof of the Butterfly Theorem on a Hyperbola

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### Abstract

*The Butterfly Theorem originally applies to a circle. Several researchers have proven the Butterfly Theorem using various methods of proof. Furthermore, the theorem has been extended from its original application on circles to other conic sections, namely the parabola and the ellipse, also through different proof techniques. In addition, the development of the Butterfly Theorem can be applied to another conic section, the hyperbola, through an analytic approach. In this paper, the author proves the Butterfly Theorem on a hyperbola using Haruki's Lemma. Haruki's Lemma was originally established for circles. Therefore, the author develops Haruki's Lemma specifically for the hyperbola, which is then used to prove the Butterfly Theorem on a hyperbola. Thus, a new proof of the Butterfly Theorem on a hyperbola can be established. This result highlights the effectiveness of geometric methods in extending classical theorems to other conic sections.*

**Keywords:** *Butterfly Theorem, hyperbola, Haruki's Lemma, circle.*

### 1. INTRODUCTION

In [1], the history of the Butterfly Theorem is explained. It was introduced by an astronomer, Sir William Herschel, to William Wallace through a letter. In the letter, a geometric configuration on a circle is described, namely: chords "AB" and  $CD$  intersect at point  $K$ . Furthermore, there are other chords,  $EF$  and  $HG$ , which also intersect at point  $K$ . Then, chords  $HF$  and  $EG$  intersect chord  $CD$  at points  $L$  and  $M$ , respectively. If point  $K$  is the midpoint of  $CD$  and  $ML$ , then the relation  $MK = LK$  holds.

Several solutions to the problem presented in the letter on the history of the Butterfly Theorem have been published in various research articles, such as in [2, 3, 4, 5, 6, 7, 8]. Furthermore, a generalization of the Butterfly Theorem developed using complex coordinates is presented in [9]. Another development is given in [10] through an analytical approach based on complex numbers, which shows a symmetry in the distances between intersection points on chords passing through the midpoint. In addition, a comprehensive study on the origin, development, and various proof methods of the Butterfly Theorem, including the introduction of its general form and extensions such as the Double Butterfly Theorem, is discussed in [11].

Reference [12] explains that the Butterfly Theorem has also been generalized to conic sections. Further developments extend the applicability of the Butterfly Theorem, which originally applies only to circles, to convex quadrilaterals, as described in [13]. Moreover, the development of the Butterfly Theorem on a hyperbola has also been studied in [14, 15].

In addition, one of the extensions of the Butterfly Theorem is the Double Butterfly Theorem, as presented in [16]. The proof of the Double Butterfly Theorem has been presented in [17] by several researchers. Furthermore, [18, 19] explain that the Double Butterfly Theorem has also been developed for other conic sections, namely the ellipse and the parabola, using Haruki's Lemma as the basis of the proof. Haruki's Lemma itself was introduced through the study of two non-intersecting chords in a circle, as explained in [20].

To date, there has been no discussion on proving the Butterfly Theorem on a hyperbola using Haruki's Lemma as the basis of the proof. Therefore, the author is interested in studying the Butterfly Theorem on a hyperbola with a proof that employs Haruki's Lemma applied to the hyperbola. In general, this study focuses on the Butterfly Theorem on a hyperbola, with the proof constructed using Haruki's Lemma adapted to the hyperbola.

## 2. REVIEW OF LITERATURE

In this section, several supporting theories related to the discussion in the subsequent section are presented. The exposition begins with the introduction of the Inscribed Angle Theorem, followed by the theorem on the intersection of two chords in a circle, and Haruki's Lemma as applied to a circle.

**2.1. The Intersecting Chords Theorem.** Before that, this subsection introduces the Inscribed Angle Theorem in a circle, as presented in [14, 15]. An inscribed angle in a circle is an angle formed by the intersection of two chords on the circumference of the circle. The following presents the Inscribed Angle Theorem in a circle along with an illustration of Theorem 1 can be seen in Figure 1.

**Theorem 1.** (Inscribed Angle in a Circle) Inscribed angles that subtend the same arc have equal measures.

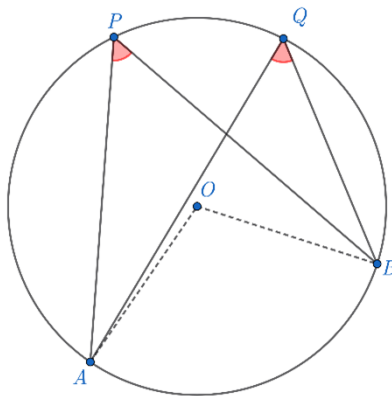


FIGURE 1. Illustration of the inscribed angle theorem in a circle

*Proof.* The proof of Theorem 1 has been presented in [14, 15]. □

Next, we discuss the Intersecting Chords Theorem. In a circle, if two chords intersect, then the products of their respective segments are equal, as explained in [14, 15]. The following presents the Intersecting Chords Theorem along with an illustration of Theorem 2 can be seen in Figure 2.

**Theorem 2.** (Intersection of Two Chords) Let  $RS$  and  $TU$  be chords of the same circle that intersect at point  $Q$ . Then, it follows that  $QR \cdot QS = QU \cdot QT$ .

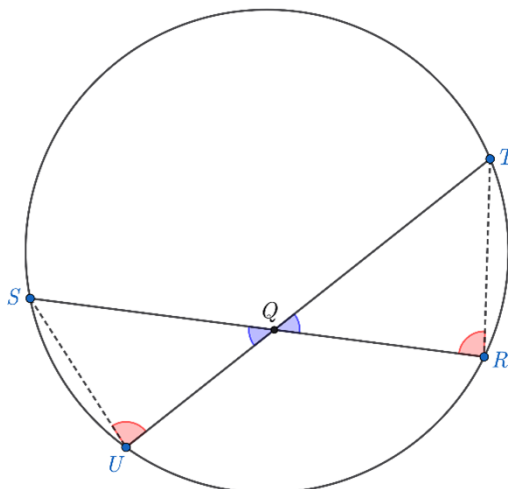


FIGURE 2. Illustration of the intersecting chords theorem

*Proof.* The proof of Theorem 2 is provided in [14, 15]. □

After discussing the intersecting chords theorem, the next discussion focuses on Haruki’s Lemma, which is related to this concept, along with its proof.

**2.2. Haruki’s Lemma.** Haruki’s Lemma is a lemma concerning two chords in a circle that do not intersect. By drawing lines from the endpoints of each chord to a point on the circle, these lines intersect the other chord at two distinct points, thereby forming a triangle. In [20], Haruki’s Lemma and its proof are explained. The following presents Haruki’s Lemma.

**Lemma 3.**(Haruki’s Lemma) Given two chords  $AB$  and  $CD$  that do not intersect in a circle, and a point  $P$  located on arc  $AB$  that does not contain points  $C$  and  $D$ . Let  $E$  and  $F$  be the intersection points of  $PC$  with  $AB$  and  $PD$  with  $AB$ , respectively. Then, the value of  $(AE \cdot BF)/EF$  is constant, that is, it does not depend on the position of point  $P$ .

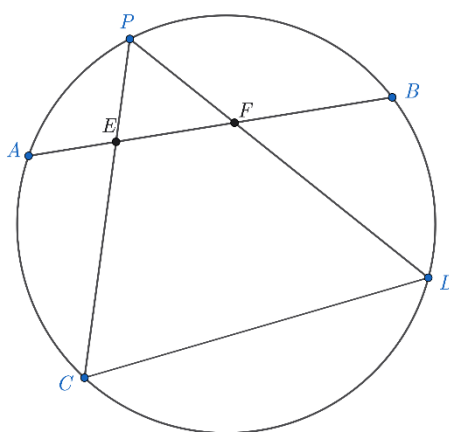


FIGURE 3. Illustration of Haruki’s Lemma

*Proof.* From Figure 3, draw a line from point  $E$  to  $D$  so that triangle  $\triangle PED$  is formed. Let the extension of line  $AB$  intersect the circumcircle of triangle  $\triangle PED$  at point  $G$ , as shown in Figure 4. □

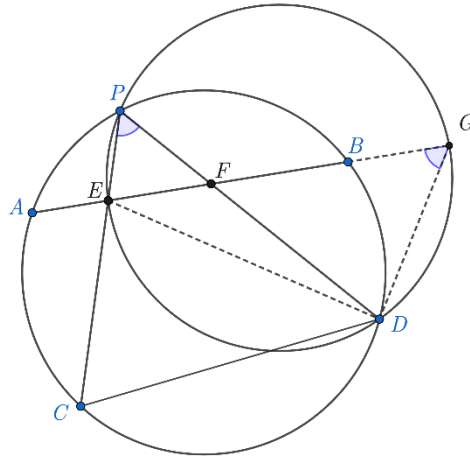


FIGURE 4. Point  $G$  remains on the extension of  $AB$

Observe that  $\angle EPD$  and  $\angle EGD$  subtend the same arc, namely arc  $ED$ . Therefore, based on Theorem 1, it follows that  $\angle EPD = \angle EGD$ . Furthermore,  $\angle EGD$  is constant for every position of point  $P$  on arc  $AB$ . Since  $\angle EGD$  is constant and point  $G$  lies on the extension of line  $AB$  intersecting the circumcircle of triangle  $\triangle PED$ , it follows that  $BG$  is also constant.

Next, observe that chords  $EG$  and  $PD$  intersect at point  $F$ . Therefore, based on Theorem 2, we obtain

$$EF \cdot FG = PF \cdot FD. \tag{1}$$

Next, chords  $AB$  and  $PD$  intersect at point  $F$ . Therefore, based on Theorem 2, we obtain

$$AF \cdot FB = PF \cdot FD. \tag{2}$$

By equating the left-hand sides of equations (1) and (2), we obtain

$$AF \cdot FB = EF \cdot FG.$$

$$(AE + EF) \cdot FB = EF \cdot (FB + BG),$$

$$AE \cdot FB + EF \cdot FB = EF \cdot FB + EF \cdot BG,$$

$$AE \cdot FB = EF \cdot BG,$$

$$\frac{AE \cdot FB}{EF} = BG. \tag{3}$$

Observe that  $BG$  does not depend on the position of  $P$ , and hence  $BG$  is constant. Consequently, the left-hand side of equation (3) is also constant. Therefore, Lemma 3 is proven.  $\square$

### 3. RESULTS AND DISCUSSION

Section 3.1 presents Haruki's Lemma on the hyperbola along with its proof, which is used to prove the Butterfly Theorem on a hyperbola. Furthermore, Section 3.2 discusses the Butterfly Theorem on a hyperbola together with its proof.

3.1. **Haruki's Lemma on a Hyperbola.** Haruki's Lemma was originally a lemma that applies to a circle. However, when the lemma is applied to a hyperbola, its property still holds. The following presents the statement of Haruki's Lemma on a hyperbola along with its proof.

**lemma 4.**(Haruki on a Hyperbola) Given two chords  $AB$  and  $CD$ , each located on the left and right sides of a hyperbola, and a point  $E$  on arc  $AB$ . Let  $F$  and  $G$  be the intersection points of the extensions of lines  $DE$  with  $AB$  and  $CE$  with  $AB$ , respectively. Then, the value of  $(AF \cdot GB)/FG$  is constant, that is, it does not depend on the position of point  $E$ .

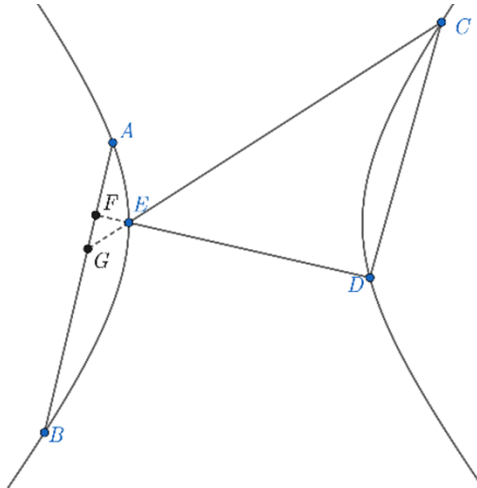


FIGURE 5. Illustration of Haruki's Lemma on a hyperbola

*Proof.* From Figure 5, construct a circle passing through points  $B, E$ , and  $A$ . Let the extensions of  $DF$  and  $CG$  intersect this circle at points  $H$  and  $I$ , respectively, as shown in Figure 6.  $\square$

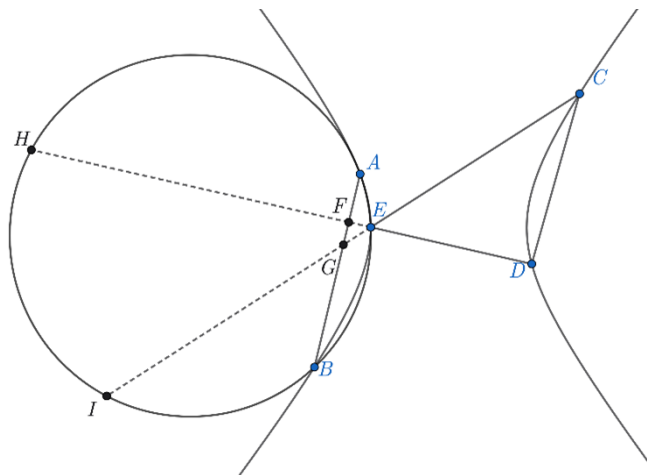


FIGURE 6. The circle passing through points  $A, E$ , and  $B$

Next, construct the circumcircle of triangle  $\triangle EFI$ . Next, construct the circumcircle of triangle  $AB$  intersect the circumcircle of triangle  $\triangle EFI$  at point  $J$ , as shown in Figure 7.

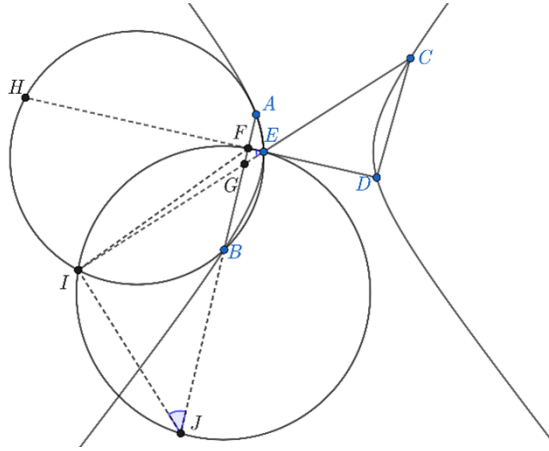


FIGURE 7. Poin  $J$  remains on the extension of line  $AB$

Observe that chords  $FJ$  and  $EI$  intersect at point  $G$ . Therefore, based on Theorem 2, we obtain

$$FG \cdot GJ = EG \cdot GI. \tag{4}$$

Next, chords  $AB$  and  $EI$  intersect at point  $G$ . Therefore, based on Theorem 2, we obtain

$$AG \cdot GB = EG \cdot GI. \tag{5}$$

By equating the left-hand sides of equations (4) and (5), we obtain

$$FG \cdot GJ = AG \cdot GB.$$

$$FG \cdot (GB + BJ) = (AF + FG) \cdot GB,$$

$$FG \cdot GB + FG \cdot BJ = AF \cdot GB + FG \cdot GB,$$

$$FG \cdot BJ = AF \cdot GB,$$

$$\frac{AF \cdot GB}{FG} = BJ. \tag{6}$$

Observe that  $BJ$  does not depend on the position of  $E$ . Therefore, based on Lemma 3, it follows that  $BJ$  is constant. Consequently, the left-hand side of equation (6) is also constant. Thus, Lemma 4 is proven.

Next, we discuss the development of the Butterfly Theorem, which originally applies to a circle and is then extended to a hyperbola, so that the theorem is referred to as the Butterfly Theorem on a hyperbola.

**3.2. The Butterfly Theorem on a Hyperbola.** The Butterfly Theorem is a theorem that applies to a circle. When the Butterfly Theorem is applied to a hyperbola, it still holds. Furthermore, the proof of the Butterfly Theorem on a hyperbola uses Lemma 4. The following presents the explanation of the Butterfly Theorem on a hyperbola along with its proof.

**theorem 5.**(Butterfly on a Hyperbola) Let  $M$  be the midpoint of chord  $AB$  on a hyperbola. Suppose there are two other chords,  $CF$  and  $ED$ . If chord  $CF$  intersects  $AB$  at  $Q$ , and chord  $ED$  intersects  $AB$  at  $P$ , then  $PM = MQ$ .

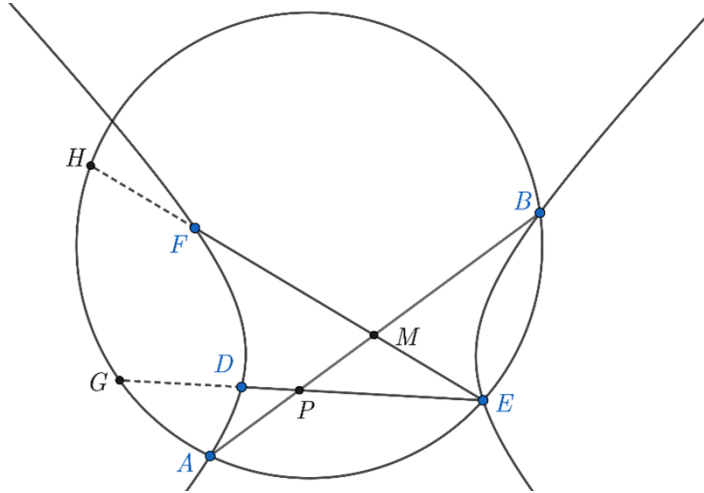


FIGURE 8. Cords  $EG$  and  $EH$  intersect at  $P$  and  $M$ , respectively

*Proof.* The proof of Theorem 5 uses Lemma 4 on chords  $EF$  and  $ED$  intersect chord  $AB$  at points  $M$  and  $P$ , respectively, as well as on chords  $CF$  and  $CD$  intersect chord  $AB$  at points  $Q$  and  $M$ , respectively. Consider Figure 8, chords  $EF$  and  $ED$  intersect chord  $AB$  at points  $M$  and  $P$ , respectively. Furthermore, by constructing a circle passing through points  $A, E$ , and  $B$ , the extensions of  $EF$  and  $ED$  intersect the circle at points  $H$  and  $G$ , respectively, as shown in Figure 9.  $\square$

Furthermore, by constructing the circumcircle of triangle  $\triangle EMG$ , and letting the extension of line  $BA$  intersect the circumcircle of triangle  $\triangle EMG$  at point  $I$ , as shown in Figure 10.

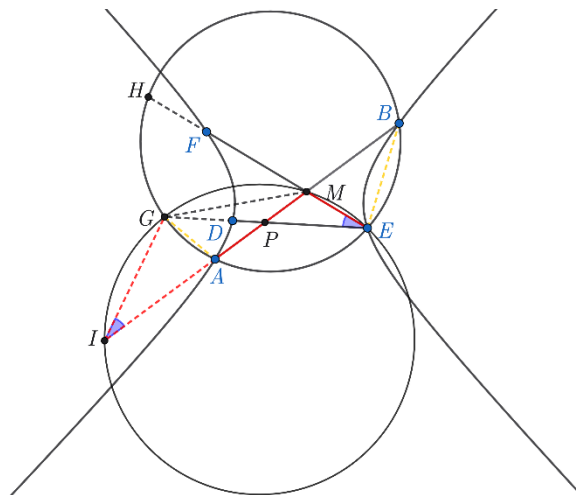


FIGURE 9. Point  $I$  remains on the extension of line  $BA$

In Figure 10, chords  $EH$  and  $EG$  intersect chord  $AB$ , so based on Lemma 4, it is obtained that

$$\frac{BM \cdot PA}{PM} = AI. \tag{7}$$

Note that  $AI$  does not depend on the position of  $E$ , so based on Lemma 4,  $AI$  is constant. Consequently, the left-hand side of equation (7) is also constant.

Consider Figure 8 chords  $CF$  and  $CD$  intersect chord  $AB$  at points  $Q$  and  $M$ , respectively. Furthermore, by constructing a circle through points  $A, C$ , and  $B$ , and letting the extensions of  $CF$  and  $CD$  intersect the circle at points  $K$  and  $J$  respectively, as shown in Figure 11.

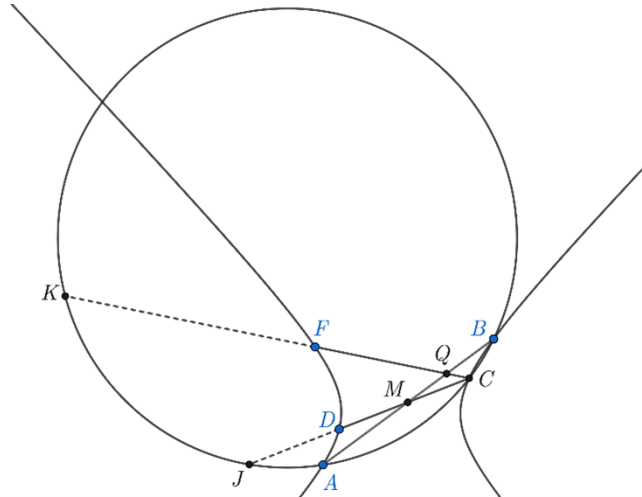


FIGURE 10. Cords  $CJ$  and  $CK$  intersect at  $M$  and  $Q$ , respectively

Furthermore, by constructing the circumcircle of triangle  $\triangle CQJ$  and letting the extension of line  $BA$  intersect the circumcircle of triangle  $\triangle CQJ$  at point  $L$ , as shown in Figure 12.

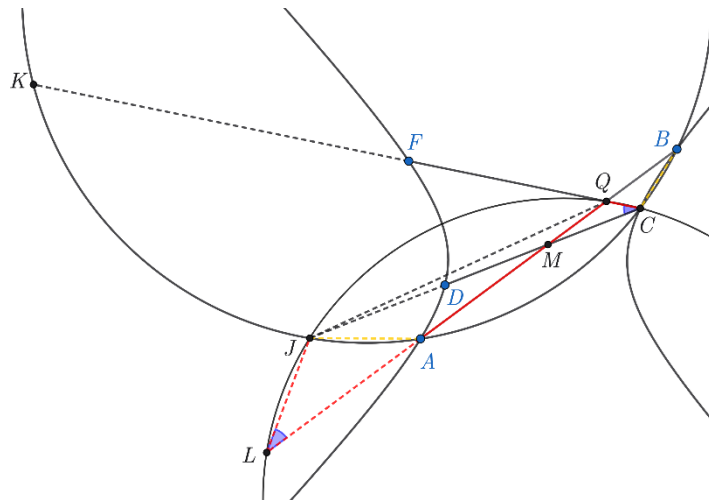


FIGURE 11. Point  $L$  remains on the extension of line  $BA$

In Figure 12, chords  $CK$  and  $CJ$  intersect chord  $AB$ , so based on Lemma 4, it is obtained that

$$\frac{BQ \cdot MA}{MQ} = AL. \tag{8}$$

Note that  $AL$  does not depend on the position of  $C$ , so based on Lemma 4,  $AL$  is constant. Consequently, the left-hand side of equation (8) is also constant. Furthermore, in Figure 8,

note that points  $C$  and  $E$  lie on the right arc of  $AB$ , so based on Lemma 3, we obtain  $AI = AL$ . By equating the left-hand sides of equations (7) and (8), it is obtained that

$$\frac{BQ \cdot MA}{MQ} = \frac{BM \cdot PA}{PM}, \quad (9)$$

By observing Figure 8, the length is obtained:

$$BM = BQ + MQ, \quad (10)$$

$$MA = PM + PA. \quad (11)$$

Since point  $M$  is the midpoint of  $AB$ , it is obtained that

$$MA = BM,$$

$$PM + PA = BQ + MQ,$$

$$BQ = PM + PA - MQ. \quad (12)$$

By substituting equations (10), (11), and (12) into equation (9), it is obtained that

$$\frac{(PM + PA - MQ)(PM + PA)}{MQ} = \frac{((PM + PA - MQ) + MQ) \cdot PA}{PM},$$

$$\frac{(PM + PA - MQ)(PM + PA)}{MQ} = \frac{(PM + PA) \cdot PA}{PM},$$

$$\frac{PM + PA - MQ}{MQ} = \frac{PA}{PM},$$

$$PM \cdot PM + PM \cdot PA - PM \cdot MQ = MQ \cdot PA,$$

$$PM \cdot PM + PM \cdot PA = MQ \cdot PA + PM \cdot MQ,$$

$$PM(PM + PA) = (PM + PA) \cdot MQ,$$

$$PM = MQ. \quad (13)$$

Based on equation (13), Theorem 5 is proven.

#### 4. CONCLUSION

Based on the discussion presented in the previous section, it can be concluded that the butterfly theorem on a hyperbola, which was originally proven using an analytic method, can also be proven using Haruki's Lemma applied to the hyperbola. The development of the butterfly theorem on a hyperbola is similar to the butterfly theorem on a circle.

#### Acknowledgement

The author would like to express sincere gratitude to the Directorate of Research, Technology, and Community Service; the Directorate General of Higher Education, Research, and Technology; the Ministry of Education, Culture, Research, and Technology of the Republic of Indonesia; and the Institute for Research and Community Service (LPPM) of Universitas Riau (grant number 29252/UN19.5.1.3/AL.04/2025) for funding this research.

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