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Convolution One Dimensional Continuous Function on Fourier Series Expansion

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Abstract
Convolution is an operation that involves two functions that can be used to transform a continuous input signal at every point in its domain so that a smooth output signal is produced at every point in the domain interval [1, 2]. But what happens when the convolution operation is applied to a function that is expanded through a Fourier series. The series is a series with a basis of differentiable functions, and how to perform convolutions that are expanded through the Fourier series. In this article, we will show a discussion to determine the product of the convolution function on the expansion of the Fourier series and the results obtained.

Keywords: *Convolution, Fourier Series, Shift Invariant Filter, Signal, Transformation.*

1. INTRODUCTION

A continuous signal is a function with one time variable which always has a real value at any time t it occupies [1]. Mathematically written

$$f(t) \in (-\infty, \infty).$$

It is well known that the convolution integral is an operation involving two continuous functions f and g which can be used to transform a continuous function into a smooth function [3]. In the library [4], the convolution integral is defined as follows.

Definition 1.1. *If f is an integrated function in the definition region $\mathbb{R}^n$ and g is a finite function that is locally integrated, then the convolution product of f and g is a function on $\mathbb{R}^n$ defined by*

$$C_g(f) = (f * g)(t) = \int_{\mathbb{R}^n} f(\tau)\, g(t-\tau)\, d\tau.$$

In the case of $n = 1$, it can be stated that if $f(t)$ and $g(t)$ are functions defined at $t \geq 0$ then the convolutions of f and g can be written by

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau, \quad t \geq 0, \tag{1}$$

provided that the integral exists.

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The function f in equation (1) is the input signal, and g is called the convolution kernel where is the dummy variable. If f is a piecewise continuous function and g is a function that is finite and disappears at the complement of the closing interval, then $(f * g)(t)$ exists for every t (see [5, 6]). More in the library [7], if $f(t)$ is differentiable where $(f * g)$ is well defined, then the convolution product is also differentiable, as $f' * g$ shown below. $(f * g)$ is differentiable and $(f * g)' = f' * g$, it can be shown as follows

$$(f * g)'(t) = \frac{d}{dt}\int f(t-\tau)\, g(\tau)\, d\tau = \int f'(t-\tau)\, g(\tau)\, d\tau = (f' * g)(t).$$

Same reason as swapping f and g, if g is differentiable then $(f * g)' = f * g'$. This statement asserts that the convolution product will be at least smooth provided that one of the convolution factors f or g is smooth for any point in the defined region.

However, in practice it does not apply to the unit ladder function f defined in $[0, 2]$ and disappears outside that interval which is convoluted by the unit ladder function g defined in the interval $[0, 1]$ and vanishes at other intervals. Consider the following convolutions f and g,

$$f(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{other} \end{cases}$$

convoluted by

$$g(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{other} \end{cases}$$

Then the convolution product of the two functions is obtained, namely

$$(f * g)(t) = \begin{cases} 0, & t < 0 \\ x, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 3 - x, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

The convolution process f by g above is depicted in Figure 1. In Figure 1.(a), it can be seen that the function f is continuous at $[0, 2]$ when g reflects and then g translates for every time t, it appears that in (b), (c), (d), and (e) the definition area of f interferes with g. As a result, there is a field bounded by the product $f(t)\, g(t-\tau)$ for each t in the definition region that is experiencing interference. Therefore the area of the intersecting plane is the convolution integral of the product. The result of the convolution is depicted in (f). It can be seen that at certain points the convolution product does not produce a smooth output.

The above case shows that the convolution product is not always able to filter the continuous function signal to always be a smooth signal at the time domain interval t. Therefore, in this article, we will show a way to determine the convolution of continuous functions f and g in order to produce a smooth output signal at the time domain interval for each t.

2. Materials and Methods

In engineering, basically any operation that maps input to output is called a filter [8]. Since most inputs and outputs are represented by functions, filters are usually map from one function space to another of the same dimensions. A filter is a linear filter if the map of this function space is linear. In practice many filtering operations are given by convolutions with fixed functions. If $\psi \in L^1(\mathbb{R}^n)$, then $C_\psi(f) = f * \psi$ defines such a filter.

A filter that takes a finite input to a finite output is called a stable filter. The estimation in equation (1) shows that each filter determined by the convolution of the one-dimensional input signal is stable (see [9]). The filter expressed by convolution has an important physical property, namely it is shift invariant [10], as stated in Proposition 2.2. The invariant shift is defined as follows,

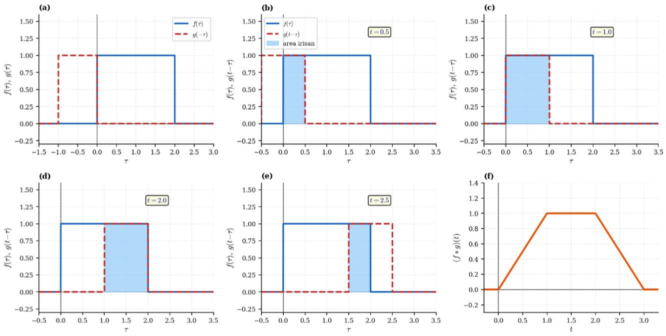

FIGURE 1. Convolution process f and g. $f(\tau) = 1_{[0,2]}$, $g(\tau) = 1_{0,1}$, horizontal axis: τ (or t), vertical axis: $f(\tau)/(f * g)(t)$.

Definition 2.1. *For $\tau \in \mathbb{R}^n$, the shift g by farτ is defined as a function g_τ by*

$$g_\tau(x) = g(x - \tau).$$

Mapping A of a function space defined on $\mathbb{R}^n$ to theother function spaces defined in $\mathbb{R}^n$ is shift invariant if it satisfies

$$A(g_\tau) = (Ag)_\tau.$$

Proposition 2.2. *Kernel in convolution is a filter function which is shift invariant*

PROOF.

$$\begin{aligned}
C_\psi(f_\tau)(x) &= \int_{\mathbb{R}^n} \psi(x-y)\, f(y-\tau)\, dy \\
&= \int_{\mathbb{R}^n} \psi(x-\tau-w)\, f(w)\, dw; \quad (w = y - \tau) \\
&= C_\psi(f)(x-\tau)
\end{aligned}$$

It is proved that the convolution kernel is shift invariant. ☐

In equation (1) it appears that for $\mathbb{R}^n$, shift operation $f \mapsto f_\tau$ is a shift invariant filter.

Based on Proposition 2.2, then for $n = 1$, the output signal of the continuous input signal f in each time domain can be determined by the weighted average of f in the interval $I \subseteq \mathbb{R}$ to a weight function ψ. The weighted average is based on the literature [11, 12, 13] expressed by equation (2), namely

$$\frac{\int_{I\subseteq\mathbb{R}} f(\tau)\psi(\tau)d\tau}{\int_{I\subseteq\mathbb{R}} \psi(\tau)d\tau} \tag{2}$$

The weighted average on equation (2) can be viewed as the convolution of the input function f with the shift invariant weight function ψ, such that $\psi_t(\tau) = g(t - \tau)$ is represented in Proposition 2.3.

Weighted average function is a function defined on $I \subseteq \mathbb{R}$ and with the weighted function we can define a scale function $\psi_\varepsilon(t)$ for a positive real number, i.e

$$\psi_\varepsilon(t) = \varepsilon^{-n}\psi\left(\frac{t}{\varepsilon}\right).$$

In this case ψ is a function that is infinitely differentiable with total integral equal to one, i.e.

$$\int_{I\subseteq\mathbb{R}} \psi(t)\, dt = 1.$$

The integral of a scale function also has a total integral equal to one. This can be shown by changing the variable $t = x$ as follows

$$\int_{I\subseteq\mathbb{R}} \psi_\varepsilon(t)\,dt = \int_{I\subseteq\mathbb{R}} \varepsilon^{-n}\,\psi\left(\frac{t}{\varepsilon}\right)\,dt = \int_{I\subseteq\mathbb{R}} \psi(x)\,dx = 1.$$

Furthermore, if f is continuous on $\mathbb{R}$ and 0, then $(\psi_\varepsilon * f)(t) \to f(t)$. It can be proved that for given $\eta > 0$ there is $\delta > 0$ such that $|t-x| < \delta \Rightarrow |f(t) - f(x)| < \eta$. This results in $|f(x)|$ limited to $x \in N_\delta(t) \subseteq \mathbb{R}$, if $\varepsilon \downarrow 0$ then

$$\begin{aligned} |(\psi_\varepsilon * f)(t) - f(t)| &= \left|\int_{N_\delta} \psi_\varepsilon(x)[f(t-x) - f(t)]\,dx\right| \\ &\leq \int_{N_\delta} |\psi_\varepsilon(x)|\,|f(t-x) - f(t)|\,dx \\ &\leq \int_{N_\delta} |\psi_\delta(x)|\,\eta dx. \end{aligned}$$

This means that it is proven that $(\psi_\varepsilon * f)(t) \to f(t)$ ☐

Proposition 2.3. *If g is a non-negative function and $\int g(\tau)\,d\tau = 1$ for each τ, then the convolutions of f and g are the weighted average and*

$$(f*g)(t) = \frac{1}{2a}\int_{t-a}^{t+a} f(\tau)d\tau.$$

Proof. Since g is a non-negative function with $\int g(\tau)d\tau = 1$, then according to equation (2) the convolution of f and g expressed by

$$(f*g)(t) = \int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau$$

is the weighted average of f for each $t \in I$, with a weight function $\psi(\tau) = g(t-\tau)$. If $g(\tau) = 0$ for $|\tau| > a$, so $g(t-\tau) = 0$ for $|t-\tau| > a$, so $(f*g)(t)$ can be viewed as the weighted average of f over the interval $[t-a, t+a]$. Especially if

$$g(t) = \begin{cases} (2a)^{-1}, & \text{if} -a < t < a \\ 0, & \text{other} \end{cases} \tag{3}$$

then get

$$(f*g)(t) = \frac{1}{2a}\int_{t-a}^{t+a} f(\tau)\,d\tau, \tag{4}$$

where the rectangular weight function $g(x)$ used for the mean integral is explicitly defined as: $g(x) = \frac{1}{2a}$ for $|x| \leq a$, and $g(x) = 0$ for $|x| > a$. This weight function ensures a uniform average over the interval. Equation 4 is therefore f over in the interval $[t-a, t+a]$. ☐

Therefore, consider the input signal f which is numerically approximated by the expansion of the function f into the Fourier series. Where the series is a representation of the reconstructed function through the Fourier coefficient sequence [14]. Before using equation (4), in this case the function to be approximated is a function $f(t)$ defined on the real line such that $f(t+2L) = f(t)$ for all $t \in [-L, L]$. The series is defined as follows

Definition 2.4. *Let the real function f be a periodic function with a period of $2L$ and integral over $[-L, L]$. The Fourier series of f is defined by*

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kt + b_k \sin kt\right), \tag{5}$$

with the Fourier coefficients a_k and b_k determined by

$$a_k = \frac{1}{L}\int_{-L}^{L} f(t)\cos kt\,dt, \quad k = 0, 1, 2, \ldots,$$

$$b_k = \frac{1}{L}\int_{-L}^{L} f(t)\sin kt\, dt, \quad k = 1, 2, \ldots$$

So that the product of the convolution of the continuous input signal always produces a smooth output signal for each point in the time domain, the convolutions of f and g in the case of the unit ladder function above can be determined by equation (4). In this case the function f is expanded first through (5) and then convoluted with equation (3) through (4).

3. Results and Discussion

Let $f_N(t)$ be the partial sum of f expressed in equation (5) such that,

$$f_N(t) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kt + b_k \sin kt)\,.$$

If $f_N(t)$ is convoluted by equation (3) through equation (4), then according to [15] obtained the following results

$$(f_N * g)(t) = \frac{1}{2a}\int_{t-a}^{t+a} f_N(\tau)\, d\tau. \tag{6}$$

In [16] it is stated that $f_N(t) \to f(t)$ for $N \to \infty$. In this case $(f_N * g)(t)$ is the average of the oscillatory function $f_N(\tau)$ centered at point t. So if $a = \frac{\pi}{N}$, then equation (6) can be written as

$$(f_N * g)(t) = \frac{N}{2\pi}\int_{t-\frac{\pi}{N}}^{t+\frac{\pi}{N}} f_N(\tau)\, d\tau\ ; \qquad \text{with } g(t) = \frac{N}{2\pi}$$

so

$$\begin{aligned}
(f_N * g)(t) &= \frac{N}{2\pi}\int_{t-\frac{\pi}{N}}^{t+\frac{\pi}{N}} f_N(\tau)\, d\tau \\
&= \frac{N}{2\pi}\left[\frac{a_0}{2}\frac{2\pi}{N} + \sum_{k=1}^{N}\left(a_k\frac{\sin k\tau}{k} - b_k\frac{\cos k\tau}{k}\right)\Bigg|_{t-\frac{\pi}{N}}^{t+\frac{\pi}{N}}\right] \\
&= \frac{N}{2\pi}\Bigg(\frac{\pi}{N}a_0 + \sum_{k=1}^{n}\Bigg\{\frac{a_k}{k}\left[\sin k\left(t+\frac{\pi}{N}\right) - \sin k\left(t-\frac{\pi}{N}\right)\right] \\
&\qquad - \frac{b_k}{k}\left[\cos k\left(t+\frac{\pi}{N}\right) - \cos k\left(t-\frac{\pi}{N}\right)\right]\Bigg\}\Bigg) \\
&= \frac{a_0}{2} + \frac{N}{2\pi}\sum_{k=1}^{N}\frac{2a_k}{k}\sin\left(\frac{\pi k}{n}\right)\cos(kt) + \frac{2b_k}{k}\sin\left(\frac{\pi k}{n}\right)\sin(kt) \\
&= \frac{a_0}{2} + \sum_{k=1}^{N}\frac{\sin\left(\dfrac{\pi k}{N}\right)}{\dfrac{\pi k}{N}}(a_k\cos kt + b_k \sin kt)
\end{aligned}$$

so that the form

$$(f_N * g)(t) = \frac{a_0}{2} + \sum_{k=1}^{N}\sigma_k(a_k\cos kt + b_k\sin kt), \tag{7}$$

where $\sigma_k = \dfrac{\sin\left(\dfrac{\pi k}{N}\right)}{\dfrac{\pi k}{N}}$.

Equation (7) is a one-dimensional convolution of the input signal of the Fourier series. Note that with the factor σ_k that arises (known as the *Sinc Filter* or *Lanczos Sigma Factor*), the Fourier series in equation (5) is modified into equation (7), and a fact is obtained that equation (7) can be viewed as the product of the convolution between the partial sum of the Fourier series and equation (3). So that the convolutions of f and g from the example above can be determined by equation (7). Equation (8) below is the result of the convolution of the illustration of the unit function $f(t)$ in the introduction above.

$$(f_N * g)(t) = \frac{1}{2} + \sum_{k=1}^{N} \frac{\sin\left(\dfrac{(2k+1)\pi}{N}\right)}{(2k+1)\dfrac{\pi}{N}} \frac{2}{(2k+1)\pi} \sin(2k+1)\pi t \tag{8}$$

The convolution product of the input signal f on the Fourier expansion through equation (8) is described as follows.

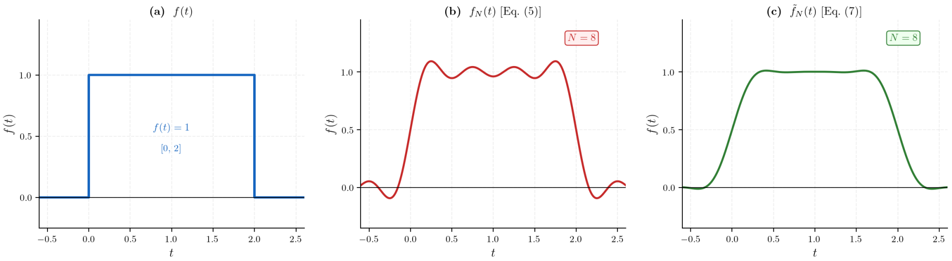

FIGURE 2. *Convolution f in Fourier Expansion.* Panel (a): original $f(t) = 1$ on $[0, 2]$, $f(t) = 0$ elsewhere;panel (b):$f_N(t)$ partial Fourier sum, $N = 8$ [Eq. (5)]; panel (c): sinc-filtered result $\tilde{f}_N(t)$, $N = 8$ [Eq. (7)]. Horizontal axis: t, vertical axis: $f(t)$. Domain : $L = 2$, period$= 2L - 4$.

In Figure 2.(a) it seems that the input signal is unit $f(t)$ before being convoluted. In Figure 2.(b) the expanded input signal f is the partial sum $f_N(t)$ of the Fourier series. Figure 2.(c) the result of the convolution of f through equation (8). The convolution results show that in the Fourier series expansion, the input signal f undergoes filtering so that f is not only smooth but the noise that occurs in the expansion of the input signal is damped, and convergence can be achieved in a relatively short time. Notably, the standard Fourier coefficients decay at a rate of $O(\frac{1}{k})$, whereas the application of the Sinc weight factor $\frac{\sin(ka)}{ka}$ results in faster decay of higher-frequency components, which directly explains the improved convergence observed here. The panels in 2 illustrate this process: panel (a) shows the original discontinuous signal $f(t)$ on the horizontal axis t before convolution; panel (b) displays the expanded input signal as a partial sum of the Fourier series with high-frequency oscillations visible on both the horizontal axis t and vertical axis $f(t)$; and panel (c) demonstrates the smoothed output after convolution via equation (8), where the Sinc Filter has suppressed the high-frequency noise. The following are some of the results of the one-dimensional $f(t)$ input signal convolution obtained through equation (7).

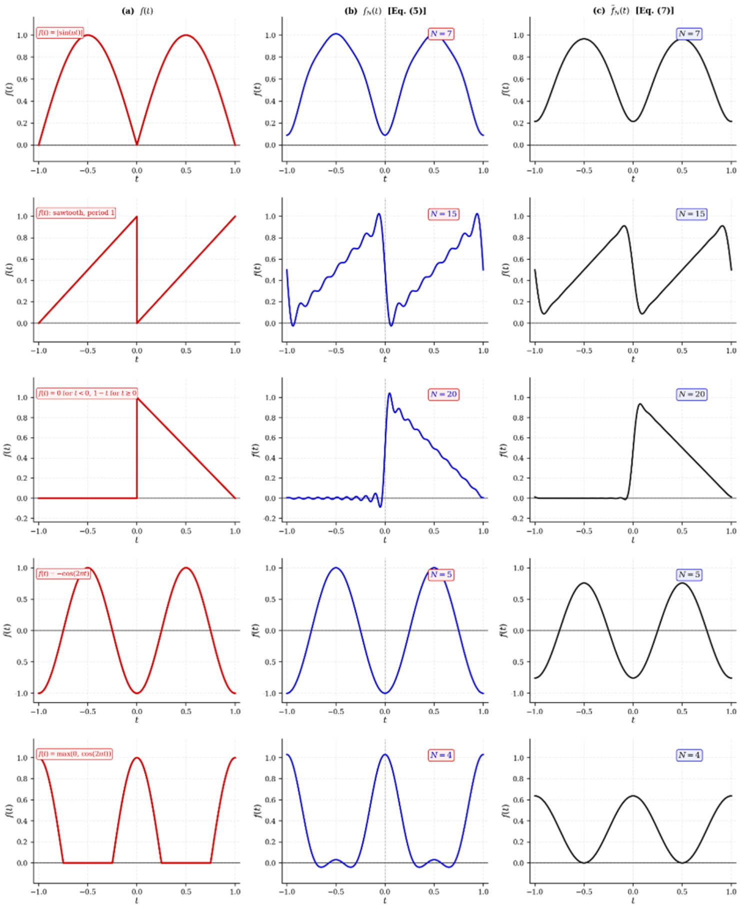

FIGURE 3. *Input Signal Convolution in Fourier Expansion.* Column (a): original signal $f(t)$; Column (b): Fourier series partial sum $f_N(t)$ [Eq. (5)]; Column (c): sinc-weighted convolution result $\tilde{f}_N(t)$ [Eq. (7)]. Horizontal axis: t, vertical axis: $f(t)$.

4. CONCLUSION

It is a fact that convolution is a weighted average of the continuous input signal f in each time domain t. The weight function ψ is the kernel of the convolution integral which is shift invariant such that $\psi(\tau) = g(t - \tau)$. The weight function in the convolution of a continuous function expanded through a Fourier series is always constant and has a total integral equal to one. The convolution of the function through Fourier expansion is the product of the convolution of the partial sum of the Fourier series with the weight function. The product of the convolution is a modified Fourier series containing the σ_k factor. This σ_k factor can reduce the jump of the input signal (*Gibbs phenomenon*) which is expanded through the Fourier series. More precisely, the σ_k factor, known as the Sinc Filter or Lanczos Sigma Factor, acts as a low-pass filter: by attenuating higher-frequency harmonics (where k is large), it suppresses the "ringing" artifacts near discontinuities. This explains why the convolution effectively reduces the Gibbs phenomenon. The product of the convolution of a continuous function expanded through a Fourier series always produces a smooth function at every point in its definition interval with a relatively short convergence time. So that the convolution product of the one-dimensional input signal can produce a smooth output signal on the same dimension for each time domain t, and can reduce noise.

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Conflict of Interest. The authors confirm that there is no conflict of interest with any parties in this research study.

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